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On Orthogonal Polynomials and the Malliavin Derivative for Lévy Stochastic Measures

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Abstract

We consider an orthogonal system of stochastic polynomials with respect to a Lévy stoachastic measure on a general topological space. In the case the stochastic measure is Gaussian or of the Poisson type, this orthogonal system turns out to have properties similar to the ones of the Hermite polynomials of Gaussian variables. In this paper we also deal with stochastic differentiation with respect to Lévy stochastic measures on topological spaces. We introduce a version of the Malliavin derivative and we suggest a direct differentiation formula which is valid for all stochastic polynomials.

Key words: Lévy stochastic measure, orthogonal multilinear form, Hermite polynomial, stochastic polynomial derivative, Malliavin derivative.

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1 Introduction.

For a probability space $(\Omega, \mathfrak{A}, \mathcal{P})$, let $L_2(\Omega)$ be the standard L_2-space of real random variables $\xi := \xi(\omega)$, $\omega \in \Omega$, with the norm

$$\|\xi\| = \Big(\int_\Omega \xi(\omega)^2 \mathcal{P}(d\omega) \Big)^{1/2}.$$

Let the σ-algebra $\mathfrak{A}$ of the events $A \subseteq \Omega$ be generated by the values $\mu(\Delta)$, $\Delta \subseteq X$, of the *stochastic measure* $\mu(dx)$, $x \in X$, on the *separable topological space* X equipped with a *tight* σ-finite Borel measure $\mathcal{M}(dx)$, $x \in X$, with *no atoms*. Here the values $\mu(\Delta)$, $\Delta \subseteq X$, are defined for all Borel sets Δ: $\mathcal{M}(\Delta) < \infty$, in such a way that

$$\mu(\Delta_1 \cup \Delta_2) = \mu(\Delta_1) + \mu(\Delta_2)$$

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for the *disjoint* sets $\Delta_1, \Delta_2 \subseteq X$. The values $\mu(\Delta)$, $\Delta \subseteq X$, are random variables in $L_2(\Omega)$, *independent* on *disjoint* sets. And these random variables obey the deFinetti-Kolmogorov-Lévy-Khintchine infinitely divisible law

$$\log Ee^{iu\mu(\Delta)} = \mathcal{M}(\Delta)\Big\{-\frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} \big(e^{iur} - 1 - iur\big)\mathcal{N}(dr)\Big\}, \quad u \in \mathbb{R} \tag{1.1}$$

- cf. e.g. [deF], [K], [I] and also [B], [Sa]. We refer to $\mu(dx)$, $x \in X$, as the *Lévy stochastic measure*. Note that $E\mu(\Delta) = 0$ and

$$E\mu(\Delta)^2 = \mathcal{M}(\Delta)\Big[\sigma^2 + \int_{\mathbb{R}} r^2 \mathcal{N}(dr)\Big]. \tag{1.2}$$

In the sequel we assume that

$$\sigma^2 + \int_{\mathbb{R}} r^2 \mathcal{N}(dr) = 1$$

in order to simplify the notations.

We suppose that the σ-finite Borel measure $\mathcal{N}(dr)$, $r \in \mathbb{R}$ with $\mathbb{R} = (-\infty, 0) \cup (0, \infty)$, in the law (1.1) satisfies the condition

$$\int_{\mathbb{R}} \big(e^{ur} - 1 - ur\big)\mathcal{N}(dr) < \infty, \quad u \in \mathbb{R}. \tag{1.3}$$

This implies that all the polynomials of the values $\mu(\Delta)$, $\Delta \subseteq X$, are elements of $L_2(\Omega)$. Moreover the polynomials are *dense* in this L_2-space.

According to (1.1) the Lévy stochastic measure is *Gaussian* if $\mathcal{N}(dr) \equiv 0$ and it is of the Poisson type if $\sigma^2 = 0$ and $\mathcal{N}(dr)$ is concentrated in some single point $\rho \neq 0$. In the latter case

$$\mu(\Delta) = \rho\big[\nu(\Delta) - E\nu(\Delta)\big], \quad \Delta \subseteq X, \tag{1.4}$$

for the Poisson random variables $\nu(\Delta)$, $\Delta \subseteq X$.

We recall that for the *Gaussian stochastic measure* $\mu(dx)$, $x \in X$, there exists a *complete system in* $L_2(\Omega)$ constituted by the well-known *Hermite polynomials* - cf. [R], [T], for example.

The *p-order* Hermite polynomials

$$\xi = F(\xi_1, ..., \xi_m) \tag{1.5}$$

of the given group of variables $\xi_j = \mu(\Delta_j)$, $j = 1, ..., m$, are characterized by their being orthogonal in $L_2(\Omega)$ to *all* the polynomials of *all* the variables $\mu(\Delta)$, $\Delta \subseteq X$, of order *less* than p. Note that any Hermite polynomial (1.5) is *uniquely* determined as the final element, say, in the standard orthogonalization of the polynomials of the involved values $\xi_j = \mu(\Delta_j)$, $j = 1, ..., m$, of order less than or equal to p. Therefore the orthogonality of the Hermite polynomials (1.5) of order p to *all* the polynomials of *all* the values $\mu(\Delta)$, $\Delta \subseteq X$, of order less than p holds only thanks to some very specific properties of the random variables $\mu(\Delta)$, $\Delta \subseteq X$.

If we try to obtain a complete system of polynomials of the values $\xi_j = \mu(\Delta_j)$ $(j = 1, ..., m)$, where $\mu(dx)$ is a general Lévy stochastic measure on X and with the same orthogonality properties that the Hermite polynomials have, then we find that this is possible *only* for the Gaussian and the Poisson type stochastic measures. And this is because those specific properties of the random variables $\mu(\Delta)$, $\Delta \subseteq X$, are distinctive only of these types of Lévy stochastic measures. See Theorem 2.1 and Corollary 2.1.

In this paper we also consider a version of the well-known Malliavin derivative (cf. e.g. [M] and olso [N], [Ø]) for general Lévy stochastic measures of the type (1.1)-(1.3). See Theorem 4.1.

Interested by the fundamental use of polynomials both in classical and stochastic analysis, we are concerned whether the above derivative is well defined on *all* polynomials of the values of a general Lévy stochastic measure. Actually we prove that the derivative exists for *all* such polynomials if and only if the involved Lévy stochastic measure is either of the Gaussian or of the Poisson type. See Theorem4.1.

Since, in these latter two cases, any polynomial ξ of the variables $\mu(\Delta)$, $\Delta \subseteq X$, admits representation in the form (1.5) via the values $\mu(\Delta_j)$ on some given group of *disjoint* sets Δ_j, $j = 1, ..., m$, then the Malliavin derivative $D\xi$ can be accordingly determined by the following differentiation formula

$$D\xi = \sum_{j=1}^{m} \Big[\sum_{k \geq 1} \frac{\rho^{k-1}}{k!} \frac{\partial^k}{\partial \xi_j^k} F(\xi_1, ..., \xi_m) \Big] 1_{\Delta_j}(x), \qquad x \in X. \tag{1.6}$$

Note that this formula holds in both the case $\mu(dx)$, $x \in X$, is Gaussian $(\rho = 0)$ and it is of the Poisson type $(\rho \neq 0)$. Here 1_Δ stays for the indicator of the involved sets Δ. See Corollary 3.1 and Theorem 4.1.

The paper is organized as follows. In Section 2 we introduce an orthogonal system of stochastic polynomials (see Definition 2.1) which turns out to have properties similar to the ones of the Hermite polynomials of Gaussian variables if the stochastic measure involved is Gaussian or of the Poisson type. This polynomials prove to be an efficient tool for the development of the stochastic calculus that follows. In Section 3 we suggest a direct stochastic differentiation formula valid for all polynomials. See Theorem 3.1. The formula can then be specified for particular Lévy stochastic measures such as the Gaussian and the Poisson type. In Section 4 we introduce a version of the Malliavin derivative for Lévy stochastic measures.

2 Orthogonal multilinear polynomials.

For a general Lévy stochastic measure $\mu(dx)$, $x \in X$, we consider the following definitions.

Definition 2.1. The *p-order multilinear forms* of the values $\mu(\Delta)$, $\Delta \subseteq X$, on the *disjoint* sets Δ with $\mathcal{M}(\Delta) < \infty$, are the random variables ξ of the type

$$\xi := F(\xi_1, ..., \xi_p) = \prod_{j=1}^{p} \xi_j \tag{2.1}$$

where $\xi_j = \mu(\Delta_j)$, $j = 1, ..., p$, are the values of the stochastic measure on the corresponding *disjoint* sets in X. A *general multilinear form* is a linear combination of the above type random variables ξ involving *different* groups of sets Δ_j, $j = 1, ..., p$ and values of $p = 0, 1, ...$ ($p = 0$ refers to the *constants* ξ).

Definition 2.2. By *δ-partitions* of X we mean the series of *disjoint* sets

$$\Delta_{kn}, \qquad k = 1, ..., K_n \quad (n = 1, 2, ...)$$

which are, each one, constituted by the sets of all the preceeding series of partition sets, with

$$\delta_n := \max_{k=1,...,K_n} \mathcal{M}(\Delta_{kn}) \longrightarrow 0, \quad n \to \infty, \tag{2.2}$$

and such that the whole family of all the sets in all the series constitute a semi-ring generating the Borel σ-algebra in

$$X = \lim_{n\to\infty} \sum_{k=1}^{K_n} \Delta_{kn}.$$

For $p = 1, 2, ...$, let us consider the multilinear forms which involve sets Δ_j, $j = 1, ..., p$, selected from the same series of δ-partitions of X. We note that in case the random variables $\xi_j = \mu(\Delta_j)$, $j = 1, ..., p$, are the values of the stochastic measure on *different* groups of (disjoint) sets Δ_j, $j = 1, ..., p$, of the same series, then we have that the *different* forms (2.1) are *orthogonal* in $L_2(\Omega)$. To explain, for the different groups $\{\Delta'_i, i = 1, ..., p'\}$ and $\{\Delta''_j, j = 1, ..., p''\}$, with $p'' \geq p'$ say, there is at least one set Δ''_k which differs from all the sets $\Delta'_j, j = 1, ..., p'$, and therefore

$$E\Big[\prod_{i=1}^{p'} \mu(\Delta'_i) \cdot \prod_{j=1}^{p''} \mu(\Delta''_j)\Big] = E\Big[\prod_{i} \mu(\Delta'_i) \cdot \prod_{j\neq k} \mu(\Delta''_j)\Big] \cdot E\mu(\Delta''_k) = 0.$$

Namely, the corresponding multilinear forms are orthogonal.

Notation. We write $H^{(p,n)}$ for the subspace in $L_2(\Omega)$ having *orthogonal basis* represented by all the p-order multilinear forms (2.1) related to the sets of the δ-partitions of X. Note that in view of the above argument, we have that the subspaces $H^{(p,n)}$, $p = 0, 1, ...$, are orthogonal.

Definition 2.3. For the orthogonal subspaces $H^{(p,n)}$, $p = 0, 1, ...$, we apply the following limits

$$(2.3) \qquad H := \lim_{n\to\infty} \sum_{p=0}^{\infty} \oplus H^{(p,n)} = \sum_{p=0}^{\infty} \oplus H^{(p)}, \qquad H^{(p)} := \lim_{n\to\infty} H^{(p,n)}$$

which represent the corresponding subspaces in $L_2(\Omega)$.

Remark 2.1. Actually *all* the p-order multilinear forms (2.1) are elements of $H^{(p)}$:

$$(2.4) \qquad \xi = \prod_{j=1}^{p} \mu(\Delta_j) \in H^{(p)},$$

whatever the *disjoint* sets Δ_j, $j = 1, ..., p$, be in X.

Proof. In fact, in general, for any two groups of disjoint sets $\{\Delta_j',\ j = 1, ..., p\}$ and $\{\Delta_j'',\ j = 1, ..., p\}$, we have that

$$(2.5) \qquad \Big\| \prod_{j=1}^{p} \mu(\Delta_j') - \prod_{j=1}^{p} \mu(\Delta_j'') \Big\| \le const \max_{1\le j\le p} \mathcal{M}(\Delta_j' \circ \Delta_j'').$$

And this can be applied to the two groups represented by the *disjoint* Borel sets $\{\Delta_j, j = 1, ..., p\}$ in X and their *disjoint* approximating sets $\{\Delta_j^n,\ j = 1, ..., p\}$ taken from the ring of the finite unions of sets from the same n^{th}-series of δ-partitions:

$$\Delta_j = \lim_{n\to\infty} \Delta_j^n, \quad \text{i.e.} \qquad \mathcal{M}(\Delta_j \circ \Delta_j^n) \longrightarrow 0, \quad n \to \infty.$$

This shows in particular that any p-order multilinear form $\xi = \prod_{j=1}^{p} \mu(\Delta_j)$ is the limit $\xi = \lim_{n\to\infty} \xi_n$ in $L_2(\Omega)$ of the finite linear combinations $\xi_n = \prod_{j=1}^{p} \mu(\Delta_j^n)$ of p-order multilinear forms in the orthogonal basis of $H^{(p,n)}$. □

Hence we can also conclude that the limit subspaces (2.3) do *not* depend on the choice of the applied δ-partitions.

Notation. In the sequel, in relation to (2.3), we will focus on

$$(2.6) \qquad H_q := \sum_{p=0}^{q} \oplus H^{(p)}, \qquad H = \lim_{q\to\infty} H_q$$

where H_q is the closure in $L_2(\Omega)$ of all the multilinear forms of order less than or equal to q ($q = 0, 1, ...$).

Remark 2.2. For the σ-algebras $\mathfrak{A}_\Delta$, $\Delta \subseteq X$, of the events generated by the stochastic measure $\mu(dx)$, $x \in \Delta$, on $\Delta \subseteq X$, we have

$$(2.7) \qquad E\Big[\prod_{j=1}^{p} \mu(\Delta_j) | \mathfrak{A}_\Delta\Big] = \prod_{j=1}^{p} \mu(\Delta_j \cap \Delta).$$

Hence any $\mathfrak{A}_\Delta$-*measurable element* $\xi \in H_q$ can be represented as a limit

$$\xi = \lim_{n\to\infty} \xi_n, \quad \text{i.e.} \qquad \|\xi - \xi_n\| \longrightarrow 0, \quad n \to \infty,$$

of linear combinations $\xi_n \in H_q$ $(n = 1, 2, ...)$ of p-order $(p \leq q)$ multilinear forms (2.1) of the values $\mu(\Delta_j^n)$, $j = 1, ..., p$, where $\Delta_j^n \subseteq \Delta$, $j = 1, ..., p$. And this in particular implies that for the $\mathfrak{A}_{\Delta'}$-measurable $\xi' \in H_{q'}$ and the $\mathfrak{A}_{\Delta''}$-measurable $\xi'' \in H_{q''}$, where Δ' and Δ'' are disjoint, it is

$$\xi' \cdot \xi'' \in H_{q'+q''}. \tag{2.8}$$

Consequently we have that

$$\xi' \cdot \xi'' \in H$$

for $\xi', \xi'' \in H$ measurable with respect to $\mathfrak{A}_{\Delta'}, \mathfrak{A}_{\Delta''}$, whatever the *disjoint* sets $\Delta', \Delta'' \subseteq X$ be.

Definition 2.4. We consider a $p$olynomial

$$\xi = F(\xi_1, ..., \xi_m) \tag{2.9}$$

of the variables $\xi_j = \mu(\Delta_j)$, $j = 1, ..., m$. We say that it is a *polynomial of the Hermite type* if, being itself of order p, it is *orthogonal* to *all* the polynomials of *all* the values $\mu(\Delta)$, $\Delta \subseteq X$, which have order *less* than p.

Theorem 2.1. *In the case $\mu(dx)$, $x \in X$, is either Gaussian or of the Poisson type, then, for all $p = 0, 1, ...,$ all the p-order polynomials of the values $\mu(\Delta)$, $\Delta \subseteq X$, belong to the subspace H_p which is the closure of the multilinear forms of the order less than or equal to p - cf.* (2.6).

Proof. Any p-order polynomial ξ is a linear combination of the products

$$\prod_{j=1}^{m} \mu(\Delta_j)^{p_j} \quad \text{with} \qquad \sum_{j=1}^{m} p_j \leq p, \tag{2.10}$$

for any borel set $\Delta_j \subseteq X$: $\mathcal{M}(\Delta_j) < \infty$ $(j = 1, ..., m)$. We proceed by induction on p. Theorem 2.1 holds for the polynomials of order $p = 0, 1$. Assuming that this theorem holds for the polynomials of order less than p, for some $p > 1$, let us show that it also holds for the polynomials of order p. By the induction hypothesis all the products (2.10) belong to H_p in the case $p_j < p$, for all $j = 1, ..., m$ - cf. Remark 2.1 and Remark 2.2. Hence it remains to show that

$$\xi = \mu(\Delta)^p \in H_p, \quad p = 2, ...,$$

for the values $\mu(\Delta)$, $\Delta \subseteq X$. Applying the δ-partitions of X (cf. Definition 2.2), we can consider the approximations of Δ by finite sums of elements of the series of the partition. Thus with no loss of generality we can here prove the statement for

$$\Delta = \sum_{k=1}^{K_n} \Delta_{kn} \quad \text{with} \quad \delta_n = \max_{k=1,\dots,K_n} \mathcal{M}(\Delta_{kn}) \longrightarrow 0, \quad n \to \infty,$$

where the disjoint sets are taken from the same series of δ-partitions. Then we can see that all the polynomials

$$\xi_n := \xi - \sum_{k=1}^{K_n} \mu(\Delta_{kn})^p = \Big[\sum_{k=1}^{K_n} \mu(\Delta_{kn})\Big]^p - \sum_{k=1}^{K_n} \mu(\Delta_{kn})^p$$

belong to H_p. This is due to the particular structure of these polynomials. And taking the sequence

$$\xi_{0n} := \sum_{k=1}^{K_n} \mu(\Delta_{kn})^p, \quad n = 1, 2, ...,$$

into account, we can also see that there exists the limit

$$\xi_0 := \lim_{n\to\infty} \xi_{0n} \tag{2.11}$$

in $L_2(\Omega)$. Here we refer to the concept of p-variation and the related results. Moreover this limit has the following form

$$\xi_0 = \begin{cases} \mathcal{M}(\Delta), & p = 2, \\ 0, & p > 2 \end{cases}$$

- in the case $\mu(dx)$, $x \in \Delta$, is *Gaussian*, and

$$\xi_0 = \rho^{p-1}\mu(\Delta) + \rho^{p-2}\mathcal{M}(\Delta)$$

- in the case $\mu(dx)$, $x \in \Delta$, is of the *Poisson type.* To explain this latter case, we involve the Poisson stochastic measure

$$\nu(dx) := \rho^{-1}\mu(dx) + \rho^{-2}\mathcal{M}(dx), \quad x \in \Delta,$$

- cf. (1.4), and since $\delta_n \to 0$ we write

$$\xi_0 = \lim_{n\to\infty} \sum_{k=1}^{K_n} \mu(\Delta_{kn})^p = \lim_{n\to\infty} \sum_{k=1}^{K_n} \rho^p \nu(\Delta_{kn})^p = \lim_{n\to\infty} \rho^p \sum_{k=1}^{K_n} \nu(\Delta_{kn})^p 1_{\{\nu(\Delta_{kn})=0,1\}}$$

$$= \lim_{n\to\infty} \rho^p \sum_{k=1}^{K_n} \nu(\Delta_{kn}) = \rho^p \nu(\Delta) = \rho^{p-1}\mu(\Delta) + \rho^{p-2}\mathcal{M}(\Delta).$$

Finally, we remark that $\xi_0 \in H_1$ and $\xi_n \in H_p$ and thus we have

$$\xi = \lim_{n\to\infty} (\xi_n + \xi_{0n}) = \lim_{n\to\infty} (\xi_n + \xi_0) \in H_p.$$

By this we end the proof. □

Corollary 2.1. *In case $\mu(dx)$, $x \in X$, is Gaussian or of the Poisson type, the p-order multilinear forms* (2.1) *represent the complete system of polynomials of the Hermite type in $L_2(\Omega)$.*

Theorem 2.2. *Let us consider the closure H of all the multilinear forms - cf.* (2.3). *The relationship*

$$H = L_2(\Omega) \tag{2.12}$$

holds if and only if the Lévy stochastic measure is Gaussian or of the Poisson type.

Proof. The part "if" is a consequence of Theorem 2.1. The part "only if" can be verified as follows. Instead of $\mu(dx)$, $x \in X$, we can consider the Lévy stochastic measure $\mu_\varphi(dt)$, $t \in [0, T]$, which appears as the standard translation

$$\mu_\varphi(B) := \mu(\Delta), \quad \Delta = \big\{x \in X : \varphi(x) \subseteq B\big\}$$

of the Lévy stochastic measure $\mu(\Delta)$, $\Delta \subseteq X$, to the stochastic measure $\mu_\varphi(B)$ on the Borel sets $B \subseteq (0, T]$ of the time interval $[0, T]$. Here a proper mapping

$$X \ni x \quad \Longrightarrow \quad \varphi(x) \in (0, T]$$

should be applied cf. [P]. This mapping indicates *all* the Borel sets $\Delta \subseteq X$ as inverse images $\Delta = \{x : \varphi(x) \subseteq B\}$ of the corresponding sets $B \subseteq [0, T]$. Regarding $\mu_\varphi(dt)$, $t \in [0, T]$, all the multilinear forms ξ: $E\xi = 0$, admit representation by the Itô stochastic integral. Hence, according to the relationship (2.12), all the elements ξ: $E\xi = 0$, in $L_2(\Omega)$ admit this type of representation. And it holds true only if $\mu_\varphi(dt)$, $t \in [0, T]$, is Gaussian or of the Poisson type - cf. [diN2]. Here we also refer to [D] and [DM]. $\square$

3 The stochastic polynomial derivative.

First of all we remark that any polynomial of the values $\mu(\Delta)$, $\Delta \subseteq X$, of the stochastic measure can be represented as a polynomial

$$\xi = F(\xi_1, ..., \xi_m) \tag{3.1}$$

of the random variables $\xi_j = \mu(\Delta_j)$, $j = 1, ..., m$, which are the values of the considered stochastic measure on some fixed *disjoint* sets Δ_j, $j = 1, ..., m$.

With respect to the Lévy stochastic measure of the type (1.1)-(1.3) we denote the moments of the measure $\mathcal{N}$ as

$$\sigma_{k+1} = \int_{\mathbb{R}} r^{k+1} \mathcal{N}(dr), \quad k > 1. \tag{3.2}$$

In the sequel we consider the standard functional space

$$L_2(\Omega \times X) \tag{3.3}$$

of the real stochastic functions $\varphi := \varphi(x)$, $x \in X$, with the norm

$$\|\varphi\|_{L_2} := \Big(\int_X \|\varphi(x)\|^2 \mathcal{M}(dx) \Big)^{1/2}.$$

And we focus our attention on the σ-algebras

$$\mathfrak{A}_{]\Delta[}, \quad \Delta \subseteq X, \tag{3.4}$$

of the events generated by the stochastic measure $\mu(dx)$, $x \in]\Delta[$, on the complements $]\Delta[:= X \setminus \Delta$ to the indicated sets Δ in X.

In the following statement the sets Δ_{kn}, $k = 1, ..., K_n$ $(n = 1, 2, ...)$, are the δ-partitions of X - cf. Definition 2.2.

Theorem 3.1. *For any polynomial ξ of the values of the stochastic measure μ, the limit*

$$\mathfrak{D}\xi := \lim_{n\to\infty} \sum_{k=1}^{K_n} E\Big(\xi \cdot \frac{\mu(\Delta_{kn})}{E\big(|\mu(\Delta_{kn})|^2 | \mathfrak{A}_{]\Delta_{kn}[}\big)} \Big| \mathfrak{A}_{]\Delta_{kn}[}\Big) 1_{\Delta_{kn}}(x), \quad x \in X, \tag{3.5}$$

exists in $L_2(\Omega \times X)$. Such a limit can be determined by the formula

$$\mathfrak{D}\xi = \sum_{j=1}^{m} \frac{\partial}{\partial \xi_j} F(\xi_1, ..., \xi_m) 1_{\Delta_j}(x) + \sum_{j=1}^{m} \Big[\sum_{k>1} \frac{\sigma_{k+1}}{k!} \frac{\partial^k}{\partial \xi_j^k} F(\xi_1, ..., \xi_m)\Big] 1_{\Delta_j}(x),\ x \in X, \tag{3.6}$$

according to the representation (3.1) *of the polynomial ξ.*

Proof. The result can be proved by the same arguments used in [diN1] and [diN2] for the *non-anticipating stochastic derivative.* □

Corollary 3.1. *The differentiation formula* (3.6) *can be specified to*

$$\mathfrak{D}\xi = \sum_{j=1}^{m} \Big[\sum_{k\geq 1} \frac{\rho^{k-1}}{k!} \frac{\partial^k}{\partial \xi_j^k} F(\xi_1, ..., \xi_m)\Big] 1_{\Delta_j}(x), \quad x \in X, \tag{3.7}$$

in the case $\mu(dx)$, $x \in X$, is either Gaussian ($\rho = 0$) or of the Poisson type ($\rho \neq 0$) - cf. (1.4).

4 The Malliavin derivative.

We recall that the Gaussian stochastic measure $\mu(dx)$, $x \in X$, on a time interval $X = [0, T]$ is characterized by the law (1.1) with $\mathcal{M}(dx) = dx$, $x \in X$, as the Lebesgue measure and $\mathcal{N}(dr) \equiv 0$. In this case the well-known *Malliavin derivative* coincides with the *minimal closed extension* of the stochastic polynomial derivative $\mathfrak{D}\xi$ (cf. (3.5)-(3.7)), initially considered just on all the polynomials of the form (3.1)

$$\xi = F(\xi_1, ..., \xi_m) \tag{4.1}$$

such that

$$\frac{\partial^k}{\partial \xi_j^k} F(\xi_1, ..., \xi_m) = 0, \quad k \neq 1 \quad (j = 1, ..., m).$$

Namely the above polynomials are actually *m-order multilinear forms* of the values $\xi_j = \mu(\Delta_j)$, $j = 1, ..., m$, on the *disjoint* sets Δ_j, $j = 1, ..., m$ - cf. (2.1). We refer to [M], [N], [Ø] for example, for the characterization of the Malliavin derivative.

Following the above scheme we suggest a version of the Malliavin derivative with respect to a general Lévy stochastic measure $\mu(dx)$, $x \in X$ - cf. (1.1)-(1.3).

Theorem 4.1. *(a) The stochastic polynomial derivative* (3.5) *on the multilinear forms* (4.1) *is a closable linear operator. We denote its minimal closed extension by $D\xi$, $\xi \in H$:*

$$D\xi = \mathfrak{D}\xi : \quad H \supseteq \xi \quad \Longrightarrow \quad D\xi \in L_2(\Omega \times X). \tag{4.2}$$

(b) The operator D is well defined on all *the polynomials of all the values of the stochastic measure $\mu(dx)$, $x \in X$, if and only if the involved stochastic measure is Gaussian or of the Poisson type. Hence in these two cases $D\xi$, as $D\xi = \mathfrak{D}\xi$, can be determined by the formula* (3.7) *provided that ξ is represented in the form* (3.1).

Consistently with the Gaussian case we call the operator D the *Malliavin derivative for Lévy stochastic measures.*

Proof. (a) On the multilinear forms (4.1), we have $D\xi = \mathfrak{D}\xi$:

$$\mathfrak{D}\xi = \sum_{j=1}^{m} \frac{\partial}{\partial \xi_j} F(\xi_1, ..., \xi_m) 1_{\Delta_j}(x), \quad x \in X \tag{4.3}$$

- cf. (3.6). For $p = 1, 2, ...$, let

$$L_2(X, H^{(p-1)}) \subseteq L_2(X, H) \subseteq L_2(\Omega \times X) \tag{4.4}$$

be subspaces in the functional L_2-space (3.3) of the stochastic functions $\varphi := \varphi(x)$, $x \in X$, taking values in the corresponding subspaces

$$H^{(p-1)} \subseteq H \subseteq L_2(\Omega)$$

- cf. (2.3). We can see that, according to the orthogonal sum $H = \sum_{p=0}^{\infty} \oplus H^{(p)}$ it is

$$L_2(X, H) = \sum_{p=1}^{\infty} \oplus L_2(X, H^{(p-1)}). \tag{4.5}$$

Note that the operator (4.2), defined by the formula (4.3), is such that

$$H^{(p)} \ni \xi_p \quad \Longrightarrow \quad D\xi_p \in L_2(X, H^{(p-1)}).$$

And moreover we have that

$$\|D\xi_p\|_{L_2} = \sqrt{p}\|\xi_p\|, \quad \xi_p \in H^{(p)}. \tag{4.6}$$

This can be easily seen via the pre-limit subspaces $H^{(p,n)}$ in (2.3). Hence, let us consider the operator D:

$$H \supseteq domD \ni \xi \quad \Longrightarrow \quad D\xi \in L_2(X, H) \tag{4.7}$$

the domain of which is constituted by all the elements $\xi \in H$: $\xi = \sum_{p=0}^{\infty} \oplus \xi_p$ such that

$$\sum_{p=0}^{\infty} p\|\xi_p\|^2 < \infty$$

- cf. (2.3) and (4.6). This operator is naturally defined as

$$D\xi := \sum_{p=1}^{\infty} \oplus D\xi_p \tag{4.8}$$

- cf. (4.5). Hence D is a *closed* linear operator and it represents the minimal closed extension of the operator (4.2).

(b) If all the polynomials of the values of the stochastic measure μ belong to the domain $domD \subseteq H$, then it is $H = L_2(\Omega)$. And this implies that the involved stochastic measure is either Gaussian or of the Poisson type. See Theorem 2.2.

Proving the converse is more delicate. Let $\mu(dx)$, $x \in X$, be Gaussian or of the Poisson type (then $H = L_2(\Omega)$). We are going to show that, for these two types of Lévy stochastic measures, all the polynomials ξ belong to $domD$ and that the formula $D\xi = \mathfrak{D}\xi$ holds true. Let us consider the approximation

$$\xi = \lim_{n\to\infty} \xi_n, \text{ i.e. } \quad \|\xi - \xi_n\| \to 0, \quad n \to \infty, \tag{4.9}$$

of ξ by a sequence of *multilinear forms* ξ_n such that

$$\mathfrak{D}\xi := \lim_{n\to\infty} \mathfrak{D}\xi_n, \text{ i.e. } \quad \|\mathfrak{D}\xi - \mathfrak{D}\xi_n\|_{L_2} \to 0, \quad n \to \infty, \tag{4.10}$$

for the stochastic polynomial derivative (3.5)-(3.7), which we remind is well defined on all polynomials. Indeed, the approximations (4.9)-(4.10) imply that the *closed* linear operator D in (4.7)-(4.8) is such that $D\xi = \mathfrak{D}\xi$ and thus that all polynomials belong to $domD$. We proceed by induction. In a trivial way, $D\xi = \mathfrak{D}\xi$ holds for the polynomials ξ of order $p = 1$ - cf. (4.3). Let us assume that this formula holds for all the polynomials of order less than p, for some $p > 1$. Then we show that it also holds for the polynomials of order p.

As a first step we consider two polynomials ξ' and ξ'' such that their representations in form (3.1) involve the sets $\Delta_i' \subseteq \Delta'$, $i = 1, ..., m'$, and $\Delta_j'' \subseteq \Delta''$, $j = 1, ..., m''$, corresponingly laying in two *disjoint* sets Δ', Δ'' (then ξ' and ξ'' are measurable with respect to $\mathfrak{A}_{\Delta'}$ and $\mathfrak{A}_{\Delta''}$ respectively - see Remark 2.2). For all couples ξ', ξ'' of this kind, the following formula

$$\mathfrak{D}(\xi' \cdot \xi'') = \mathfrak{D}\xi' \cdot \xi'' + \xi' \cdot \mathfrak{D}\xi''$$

holds true thanks to (3.5) and

$$\|\mathfrak{D}(\xi' \cdot \xi'')\|_{L_2}^2 = \|\mathfrak{D}\xi'\|_{L_2}^2 \cdot \|\xi''\|^2 + \|\xi'\|^2 \cdot \|\mathfrak{D}\xi''\|_{L_2}^2 < \infty.$$

Hence, if ξ', ξ'' admit approximations of the above type (4.9)-(4.10) by the multilinear forms ξ_n', ξ_n'' of the values of the stochastic measure in Δ', Δ'' - cf. Remark 2.2. Then this kind of approximation (4.9)-(4.10) holds also for the product $\xi' \cdot \xi''$. Indeed, it is

$$\xi' \cdot \xi'' = \lim_{n\to\infty} \xi_n' \cdot \xi_n'' \quad \text{and} \quad \mathfrak{D}(\xi' \cdot \xi'') = \lim_{n\to\infty} \mathfrak{D}(\xi_n' \cdot \xi_n'').$$

In fact we have

$$\|\xi' \cdot \xi'' - \xi_n' \cdot \xi_n''\| \leq const(\|\xi' - \xi_n'\| + \|\xi'' - \xi_n''\|) \longrightarrow 0, \quad n \to \infty,$$

and

$$\|\mathfrak{D}(\xi' \cdot \xi'') - \mathfrak{D}(\xi_n' \cdot \xi_n'')\|_{L_2} \leq$$
$$\leq const \cdot (\|\xi' - \xi_n'\| + \|\xi'' - \xi_n''\| + \|\mathfrak{D}\xi' - \mathfrak{D}\xi_n'\|_{L_2} + \|\mathfrak{D}\xi'' - \mathfrak{D}\xi_n''\|_{L_2}) \to 0, \quad n \to \infty.$$

Thanks to (4.3) it is $D(\xi_n' \cdot \xi_n'') = \mathfrak{D}(\xi_n' \cdot \xi_n'')$ and thus $D(\xi' \cdot \xi'') = \mathfrak{D}(\xi' \cdot \xi'')$ - cf. (4.8).

According to these facts above and the assumption in the induction argument on all polynomials of order less than p, we can also conclude that the formula $D\xi = \mathfrak{D}\xi$ holds for all p-order polynomials (3.1) chich appear as a linear combination of the products

$$\prod_{j=1}^{m} \mu(\Delta_j)^{p_j} \quad \text{with} \quad \sum_{j=1}^{m} p_j \leq p \;:\; p_j \leq p - 1 \quad (j = 1, ..., m).$$

Hence it only remains to show that $D\xi = \mathfrak{D}\xi$ holds for the polynomials of the type $\xi = \mu(\Delta)^p$. Here we apply the same argument as in the proof of Theorem 2.1. Namely, with no loss of generality, we consider the set

$$\Delta = \sum_{k=1}^{K_n} \Delta_{kn} \quad \text{with} \qquad \delta_n = \max_{k=1,\dots,K_n} M(\Delta_{kn}) \to 0, \quad n \to \infty,$$

where the disjoint sets are elements of the same n^{th}-series of δ-partitions. Then we can conclude that the formula $D\xi = \mathfrak{D}\xi$ holds for all the polynomials of form

$$\xi_n := \Big[\sum_{k=1}^{K_n} \mu(\Delta_{kn})\Big]^p - \sum_{k=1}^{K_n} \mu(\Delta_{kn})^p$$

thanks to the particular structure of these polynomials. Let us then consider the *linear form* ξ_0 which appear as the limit

$$\xi_0 := \lim_{n\to\infty} \xi_{0n}$$

in $L_2(\Omega)$ of the polynomials

$$\xi_{0n} := \sum_{k=1}^{K_n} \mu(\Delta_{kn})^p,$$

- cf. (2.11). The formula (3.7) applied to ξ_0 and ξ_{0n}, $n = 1, 2, ...$, shows that the polynomial derivatives $\mathfrak{D}\xi_{0n}$, $n = 1, 2, ...$, admit the limit

$$\mathfrak{D}\xi_0 = \lim_{n\to\infty} \mathfrak{D}\xi_{0n}$$

in $L_2(\Omega \times X)$. Hence, from

$$\lim_{n\to\infty} (\xi_n + \xi_0) = \lim_{n\to\infty} (\xi + \xi_0 - \xi_{0n}) = \xi,$$

we derive

$$\lim_{n\to\infty} \mathfrak{D}(\xi_n + \xi_0) = \lim_{n\to\infty} [\mathfrak{D}\xi + \mathfrak{D}\xi_0 - \mathfrak{D}\xi_{0n}] = \mathfrak{D}\xi$$

- cf. (4.9)-(4.10). By this we end the proof. □

We would like to stress that there has been a large interest on the generalization of Malliavin calculus to other integrators than the Brownian motion. In particular much of the attention has been directed towards the study of more general Lévy processes as integrators, and this is largly due to the wide possibility of application both in physics and in mathematical finance. Here we would like to mention, just as an example, the works on the Poisson process, [BC], [BGJ], [DKW], [NV], [Pi], and on more general Lévy processes, [BDLØP], [LSUV], and on pure jump Lévy processes and the related centred Poisson stochastic measures on the product $[0, T] \times \mathbb{R}$: [DØP], [Ka], [L], [ØP].

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